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Random matrix theory and discrete moments of the Riemann zeta function

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Abstract
We calculate the discrete moments of the characteristic polynomial of a random unitary matrix, evaluated a small distance away from an eigenangle. Such results allow us to make conjectures about similar moments for the Riemann zeta function, and provide a uniform approach to understanding moments of the zeta function and its derivative.

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1. Introduction

Let $\zeta(s)$ be the Riemann zeta function, and denote its non-trivial zeros by $1/2 + \mathrm{i}\gamma_n$, with $0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots$. (For simplicity, we will assume the Riemann hypothesis, which says that $\gamma_n \in \mathbb{R}$.) It is known (see, for example, Titchmarsh's book [1] for details) that if $N(T)$ is the number of zeros with $0 < \gamma_n \leqslant T$ then

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \mathrm{O}(\log T).$$

Moments of the zeta function

$$I_k(T) := \frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + \mathrm{i}t\right) \right|^{2k} \mathrm{d}t$$

have long been of interest to number theorists, with it being widely believed that

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + \mathrm{i}t\right) \right|^{2k} \mathrm{d}t \sim f_k a(k) (\log T)^{k^2}$$

with

$$a(k) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)} \right)^2 p^{-m} \tag{1}$$

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and f_k being an integer (when k is integer) divided by $(k^2)!$, whose value was unknown apart from in a few cases. The known values of f_k are $f_0 = 1$ (trivial), $f_1 = 1$ (Hardy and Littlewood [2]), and $f_2 = 1/12$ (Ingham [3]). The value $f_3 = 42/9!$ has been conjectured by Conrey and Ghosh [4] and $f_4 = 24024/16!$ is a conjecture of Conrey and Gonek [5]. We should mention that $a(k)$ given in (1) can be calculated for certain k: $a(0) = a(1) = 1$ and $a(-1) = a(2) = 6/\pi^2$.

In [6], Keating and Snaith argued that one can create a probabilistic model for the zeta function around height T using the characteristic polynomial of an $N \times N$ unitary matrix chosen according to Haar measure, when

$$N = \log \frac{T}{2\pi}.$$

Setting

$$\begin{aligned} Z_U(\theta) &:= \det(I - \mathrm{e}^{-\mathrm{i}\theta} U) \\ &= \prod_{n=1}^{N} (1 - \mathrm{e}^{\mathrm{i}(\theta_n - \theta)}) \end{aligned}$$

and defining $M_N(2k) := \mathbb{E}_N\{|Z_U(0)|^{2k}\}$ where $\mathbb{E}_N$ denotes expectation with respect to Haar measure, they found that

$$M_N(2k) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \sim \frac{G^2(k+1)}{G(2k+1)} N^{k^2} \tag{2}$$

as $N \to \infty$ for fixed k subject to $\mathrm{Re}(k) > -1/2$, where $G(\cdot)$ is the Barnes G-function.

By comparing with the known (and previously conjectured) values of f_k, they were led to the conjecture that $f_k = G^2(k+1)/G(2k+1)$.

Conjecture 1 (Keating and Snaith)**.** *For fixed $k > -1/2$, as $T \to \infty$,*

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + \mathrm{i}t\right) \right|^{2k} \mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)} a(k) \left(\log \frac{T}{2\pi}\right)^{k^2}$$

where $G(\cdot)$ is the Barnes G-function and $a(k)$ is given in (1).

Following this, Hughes *et al* [7] used the characteristic polynomial to model the discrete moments of the derivative of the zeta function,

$$J_k(T) := \frac{1}{N(T)} \sum_{0<\gamma_n \leqslant T} \left| \zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_n\right) \right|^{2k}.$$

They calculated that for fixed k subject to $\mathrm{Re}(k) > -3/2$,

$$\mathbb{E}_N \left\{ \frac{1}{N} \sum_{n=1}^{N} |Z_U'(\theta_n)|^{2k} \right\} \sim \frac{G^2(k+2)}{G(2k+3)} N^{k(k+2)}$$

as $N \to \infty$, and they used this to conjecture the leading order term in the asymptotic expansion of $J_k(T)$.

Conjecture 2 (Hughes, Keating and O'Connell)**.** *If all the zeros of the zeta function are simple, then for fixed $k > -3/2$, as $T \to \infty$,*

$$\frac{1}{N(T)} \sum_{0<\gamma_n \leqslant T} \left| \zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_n\right) \right|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a(k) \left(\log \frac{T}{2\pi}\right)^{k(k+2)}$$

where $a(k)$ is given by (1), and $G(\cdot)$ is the Barnes G-function.

Again this conjecture is found to agree with all previously known results; when $k = -1$ (a conjecture of Gonek [8]), when $k = 0$ (trivial) and when $k = 1$ (a theorem of Gonek [9] under RH). Also, extending a theorem due to Conrey *et al* [10] (recorded in this paper as theorem 6) beyond its range of (proven) applicability, conjecture 7.1 states that $J_2(T) \sim \frac{1}{1440\pi^2}\left(\log\frac{T}{2\pi}\right)^8$, which agrees perfectly with conjecture 2.

It is striking that conjectures 1 and 2 have very similar form. The purpose of this paper is to unify them as special cases of one result.

In the next section we will show

$$\mathbb{E}_N\left\{\left|Z_U\left(\theta_1+\frac{y}{N}\right)\right|^{2k}\right\} \sim \frac{G^2(k+1)}{G(2k+1)}F_k(y)N^{k^2}$$

where $F_k(y)$ is a certain function, independent of N, given in theorem 3.

We will then use this to conjecture that for $L = \frac{1}{2\pi}\log\frac{T}{2\pi}$,

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leqslant T}\left|\zeta\left(\frac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)}a(k)F_k(2\pi\alpha)\left(\log\frac{T}{2\pi}\right)^{k^2}$$

and will show, in section 3, that this conjecture contains conjectures 1 and 2 as special cases ($\alpha \to \infty$ and $\alpha \to 0$ respectively). The conjecture is found to agree with a known result of Gonek [9], and the extension of the theorem due to Conrey *et al* [10] cited above. These comparisons are discussed in section 3.1.

2. The random matrix calculation

Theorem 3. *For fixed k with* $\mathrm{Re}(k) > -1/2$, *and for* $|x| \leqslant AN$, *with* $A < \pi$ *constant,*

$$\mathbb{E}_N\left\{\left|Z_U\left(\theta_1+\frac{2x}{N}\right)\right|^{2k}\right\} = \frac{G^2(k+1)}{G(2k+1)}F_k(2x)N^{k^2} + \mathrm{O}\big(N^{k^2-1}\big)$$

where

$$F_k(2x) = x^2 j_k(x)^2 + x^2 j_{k-1}(x)^2 - 2kx j_k(x) j_{k-1}(x) \tag{3}$$

where $j_n(x)$ *are the spherical Bessel functions of the first kind.*

Proof. First note that

$$|Z_U(\theta_N+\beta)|^{2k} = |1-\mathrm{e}^{-\mathrm{i}\beta}|^{2k}\prod_{n=1}^{N-1}|1-\mathrm{e}^{\mathrm{i}(\theta_n-\theta_N-\beta)}|^{2k}.$$

The average of this over all $N \times N$ unitary matrices with Haar measure can be written [11, 12] as an N-fold integral

$$\mathbb{E}_N\left\{\prod_{n=1}^{N-1}|1-\mathrm{e}^{\mathrm{i}(\theta_n-\theta_N-\beta)}|^{2k}\right\} = \frac{1}{N!(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{1\leqslant i<j\leqslant N}|\mathrm{e}^{\mathrm{i}\theta_i}-\mathrm{e}^{\mathrm{i}\theta_j}|^2$$
$$\times\prod_{n=1}^{N-1}|1-\mathrm{e}^{\mathrm{i}(\theta_n-\theta_N-\beta)}|^{2k}\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N.$$

Putting all the $j = N$ terms from the first product into the second, we see that

$$\mathbb{E}_N\{|Z_U(\theta_N+\beta)|^{2k}\} = \frac{1}{N}\left|2\sin\left(\tfrac{1}{2}\beta\right)\right|^{2k}\mathbb{E}_{N-1}\{|Z_{\widetilde{U}}(0)|^2|Z_{\widetilde{U}}(\beta)|^{2k}\} \tag{4}$$

where $Z_{\widetilde{U}}$ is the characteristic polynomial of an $(N-1)\times(N-1)$ unitary matrix.

By rotation invariance of Haar measure,

$$\mathbb{E}_{N-1}\{|Z_{\widetilde{U}}(0)|^2|Z_{\widetilde{U}}(\beta)|^{2k}\} = \mathbb{E}_{N-1}\{|Z_{\widetilde{U}}(0)|^{2k}|Z_{\widetilde{U}}(\beta)|^2\}.$$

This is calculated in theorem 4, where it is shown that for $\mathrm{Re}(k) > -1/2$,

$$\mathbb{E}_N\{|Z_U(0)|^{2k}|Z_U(y/N)|^2\} \sim \frac{G^2(k+1)}{G(2k+1)} \sum_{p=0}^{\infty} \frac{k(k-1+p)!(k+p)!}{p!(2k+p)!(2k+1+2p)!}(-1)^p y^{2p} N^{(k+1)^2}$$

and substituting this into (4) (where we put $\beta = y/N$) we see that

$$\mathbb{E}_N\left\{\left|Z_U\left(\theta_N + \frac{y}{N}\right)\right|^{2k}\right\} = \frac{G^2(k+1)}{G(2k+1)} F_k(y) N^{k^2}\left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right)$$

where

$$F_k(y) = k\sum_{p=0}^{\infty} \frac{(k-1+p)!(k+p)!}{p!(2k+p)!(2k+1+2p)!}(-1)^p y^{2k+2p}. \tag{5}$$

The spherical Bessel functions of the first kind are defined as

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)!}{m!(2n+2m+1)!2^{2m}}(2z)^{n+2m}$$

where $J_\nu(z)$ is the νth order Bessel function of the first kind. Hence,

$$F_k(2x) = x^2 j_k(x)^2 + x^2 j_{k-1}(x)^2 - 2kx j_k(x) j_{k-1}(x)$$

which can be seen by comparing the Taylor expansions.

The above large-N asymptotics are for $x = o(N)$. This can be extended to $|x| \leqslant AN$ for an arbitrary constant $A < \pi$ as follows: let β be a fixed constant subject to $0 < \beta < 2\pi$. By (4) and the results of Basor [13],

$$\mathbb{E}_N\{|Z_U(\theta_1 + \beta)|^{2k}\} = \frac{1}{N}\left|2\sin\left(\tfrac{1}{2}\beta\right)\right|^{2k} \mathbb{E}_{N-1}\{|Z_{\widetilde{U}}(0)|^{2k}|Z_{\widetilde{U}}(\beta)|^2\} \sim \frac{G^2(k+1)}{G(2k+1)} N^{k^2}.$$

If one lets $x = N\beta/2$ then the large-x asymptotics of Bessel functions (see, for example, chapter 9 of [14]) implies that

$$F_k(N\beta) = 1 + \mathrm{O}\left(\frac{1}{N}\right)$$

and so theorem 3 gives the correct first-order term, for $|x| \leqslant AN$ for $A < \pi$ a constant. □

Remark. When n is an integer,

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{\sin x}{x}$$

which leads to a neat evaluation of $F_k(2x)$ for integer k, the first few being:

$$F_1(2x) = \frac{x^2 - \sin^2(x)}{x^2}$$

$$F_2(2x) = \frac{x^4 - 3x^2 + 3x\sin(2x) + (2x^2 - 3)\sin^2(x)}{x^4} \tag{6}$$

$$F_3(2x) = \frac{x^6 - 3x^4 - 45x^2 + (-12x^3 + 45x)\sin(2x) + (-3x^4 + 72x^2 - 45)\sin^2(x)}{x^6}.$$

Theorem 4.

$$\mathbb{E}_N\{|Z(0)|^{2k}|Z(\beta)|^2\} = M_N(2k)N!(N+2k)! \times \sum_{n=0}^{N} \frac{(2\sin(\beta/2))^{2n}}{n!(2k+n)!} \sum_{m=0}^{N-n} \frac{(N+k-m)!(k+m+n)!}{(N-n-m)!m!} \mathrm{e}^{\mathrm{i}\beta(2m-N+n)}$$

where $M_N(2k)$ is given in (2). If $\frac{y}{N} \to 0$ as $N \to \infty$, then for fixed $\mathrm{Re}(k) > -1/2$,

$$\mathbb{E}_N\left\{|Z(0)|^{2k}\left|Z\left(\frac{y}{N}\right)\right|^2\right\} = \frac{G^2(k+1)}{G(2k+1)} \sum_{p=0}^{\infty} \frac{k(k-1+p)!(k+p)!}{p!(2k+p)!(2k+1+2p)!}(-1)^p y^{2p} \times N^{(k+1)^2}\left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right).$$

Proof. If k is an integer, then

$$\mathbb{E}_N\left\{\prod_{n=1}^{N}|1-\mathrm{e}^{\mathrm{i}\theta_n}|^{2k}|1-\mathrm{e}^{\mathrm{i}\theta_n}\,\mathrm{e}^{-\mathrm{i}\beta}|^2\right\} = \frac{1}{(2\pi)^N N!}\int\cdots\int_{-\pi}^{\pi}\prod_{1\leqslant j<m\leqslant N}(\mathrm{e}^{\mathrm{i}\theta_j}-\mathrm{e}^{\mathrm{i}\theta_m})(\mathrm{e}^{-\mathrm{i}\theta_j}-\mathrm{e}^{-\mathrm{i}\theta_m}) \times \prod_{n=1}^{N}(1-\mathrm{e}^{\mathrm{i}\theta_n})^k(1-\mathrm{e}^{-\mathrm{i}\theta_n})^k(1-\mathrm{e}^{\mathrm{i}\theta_n}\,\mathrm{e}^{-\mathrm{i}\beta})(1-\mathrm{e}^{-\mathrm{i}\theta_n}\,\mathrm{e}^{\mathrm{i}\beta})\,\mathrm{d}\theta_n$$

which equals, after some simple manipulation of the terms

$$\frac{(-1)^{N(N-1)/2+kN+N}\mathrm{e}^{-\mathrm{i}N\beta}}{(2\pi)^N N!}\int\cdots\int_{-\pi}^{\pi}\prod_{1\leqslant j<m\leqslant N}(\mathrm{e}^{\mathrm{i}\theta_j}-\mathrm{e}^{\mathrm{i}\theta_m})^2 \times \prod_{n=1}^{N}(\mathrm{e}^{\mathrm{i}\theta_n})^{-N-k}(1-\mathrm{e}^{\mathrm{i}\theta_n})^{2k}(\mathrm{e}^{\mathrm{i}\theta_n}-\mathrm{e}^{\mathrm{i}\beta})^2\,\mathrm{d}\theta_n.$$

Therefore,

$$\mathbb{E}_N\{|Z_U(0)|^{2k}|Z_U(\beta)|^2\} = \frac{\mathrm{e}^{-\mathrm{i}N\beta}}{N!}(-1)^{N(N-1)/2+kN+N} \times \mathrm{CT}\left\{\prod_{1\leqslant j<m\leqslant N}(t_j-t_m)^2\prod_{n=1}^{N}\frac{1}{t_n^{N+k}}(1-t_n)^{2k}(t_n-\mathrm{e}^{\mathrm{i}\beta})^2\right\}$$

where $\mathrm{CT}\{\cdot\}$ denotes the constant term in the Laurent expansion in the variables $t_1,\ldots,t_N$. The constant term equals (by lemma 1 of [15])

$$\lim_{y\to 0} y^N\int\cdots\int_0^1\prod_{1\leqslant j<m\leqslant N}(t_j-t_m)^2\prod_{n=1}^{N}t_n^{-N-k+y-1}(1-t_n)^{2k}(t_n-\mathrm{e}^{\mathrm{i}\beta})^2\,\mathrm{d}t_n.$$

Kaneko [16] has evaluated this integral (which is a generalization of Selberg's integral) as

$$\prod_{j=1}^{N}\frac{\Gamma(1+j)\Gamma(j+y-N-k)\Gamma(j+2k+1)}{\Gamma(j+y+k+1)} \times \sum_{m,n=0}^{\infty}\frac{(-N)_{m+n}(y+k+1)_{m+n}}{(y-N-k)_m(2k+1)_n}\frac{\mathrm{e}^{2m\mathrm{i}\beta}(1-\mathrm{e}^{\mathrm{i}\beta})^{2n}}{m!n!}$$

where $(a)_n = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. Since we have assumed that k is an integer,

$$\lim_{y\to 0} y\Gamma(y+j-N-k) = \frac{(-1)^{N+k-j}}{\Gamma(N+k-j+1)}$$

and so we have

$$\mathbb{E}_N\{|Z_U(0)|^{2k}|Z_U(\beta)|^2\} = \frac{\mathrm{e}^{-\mathrm{i}N\beta}}{N!}\prod_{j=1}^{N}\frac{\Gamma(1+j)\Gamma(j+2k+1)}{\Gamma(j+k+1)\Gamma(N+k-j+1)}$$
$$\times \sum_{m,n=0}^{\infty}\frac{(-N)_{m+n}(k+1)_{m+n}}{(-N-k)_m(2k+1)_n}\frac{\mathrm{e}^{2m\mathrm{i}\beta}(1-\mathrm{e}^{\mathrm{i}\beta})^{2n}}{m!n!}.$$

Expanding everything out in terms of the gamma function

$$\mathbb{E}_N\{|Z_U(0)|^{2k}|Z_U(\beta)|^2\} = M_N(2k)\frac{\Gamma(N+1+2k)N!}{\Gamma(N+1+k)^2}\sum_{n=0}^{N}\frac{(2\sin(\beta/2))^{2n}}{n!\Gamma(2k+n+1)}$$
$$\times \sum_{m=0}^{N-n}\frac{\Gamma(N+k+1-m)\Gamma(k+n+1+m)}{\Gamma(N+1-n-m)!}\frac{\mathrm{e}^{\mathrm{i}\beta(2m-N+n)}}{m!}$$

where $M_N(2k)$ is defined in (2). Observe that the inner summand is invariant as $m \longrightarrow N-n-m$, and so the inner sum is in fact a sum of cosines (and thus the series expansion in β contains only even powers of β). Furthermore, observe that both sides of the equation are analytic functions of k (for $\mathrm{Re}(k) \geqslant 0$), both sides can be easily bounded by $\mathrm{O}(2^{2N\mathrm{Re}(k)})$ (for large k, with N and β fixed), and the two sides are equal at the positive integers. Thus by Carlson's theorem (see section 17 of [12], for example), the restriction that k must be an integer in the above calculations is no longer required, and the left-hand side equals the right-hand side for all complex k.

Now, using the fact that $(-a)_n = (-1)^n\Gamma(a+1)/\Gamma(a+1-n)$, we have

$$\sum_{m=0}^{N-n}\frac{\Gamma(N+k+1-m)\Gamma(k+n+1+m)}{\Gamma(N+1-n-m)}\frac{\mathrm{e}^{\mathrm{i}\beta(2m-N+n)}}{m!}$$
$$= \frac{\Gamma(N+k+1)\Gamma(k+n+1)}{\Gamma(N-n+1)}{}_2F_1\left(\begin{matrix}-N+n, k+n+1\\ -N-k\end{matrix}; \mathrm{e}^{2\mathrm{i}\beta}\right)\mathrm{e}^{\mathrm{i}(n-N)\beta}$$

where

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}$$

is the Gauss hypergeometric function. (If a, c are negative integers with $-a < -c$ then it is a polynomial of degree $-a$.)

Applying the quadratic hypergeometric transformation 15.3.26 of [14] we get

$${}_2F_1\left(\begin{matrix}-N+n, k+n+1\\ -N-k\end{matrix}; \mathrm{e}^{2\mathrm{i}\beta}\right)\mathrm{e}^{\mathrm{i}(n-N)\beta}$$
$$= {}_2F_1\left(\begin{matrix}-\frac{1}{2}N+\frac{1}{2}n, -\frac{1}{2}N+\frac{1}{2}n+\frac{1}{2}\\ -N-k\end{matrix}; \frac{1}{\cos^2(\beta)}\right)(2\cos\beta)^{N-n}. \tag{7}$$

For a positive integer m,

$${}_2F_1\left(\begin{matrix}-m, b\\ c\end{matrix}; z\right) = \frac{\Gamma(1-c-m)\Gamma(1-c+b)}{\Gamma(1-c)\Gamma(1-c-m+b)}{}_2F_1\left(\begin{matrix}-m, b\\ 1-c-m+b\end{matrix}; 1-z\right)$$

and so we see that the right-hand side of (7) equals

$$\frac{\Gamma\left(\frac{1}{2}N+\frac{1}{2}n+k+1\right)\Gamma\left(\frac{1}{2}N+\frac{1}{2}n+k+\frac{3}{2}\right)}{\Gamma(N+k+1)\Gamma\left(k+n+\frac{3}{2}\right)}(2\cos\beta)^{N-n} \times {}_2F_1\left(\begin{matrix}-\frac{1}{2}N+\frac{1}{2}n, -\frac{1}{2}N+\frac{1}{2}n+\frac{1}{2}\\ n+k+\frac{3}{2}\end{matrix}; 1-\frac{1}{\cos^2(\beta)}\right).$$

Therefore, we have proven

$$\mathbb{E}_N\{|Z_U(0)|^{2k}|Z_U(\beta)|^2\} = \sum_{n=0}^{N}\sum_{m=0}^{\lfloor\frac{1}{2}(N-n)\rfloor} T(N,k,m,n)(-1)^m(2\sin(\beta/2))^{2n}(\sin\beta)^{2m}(\cos\beta)^{N-n-2m}$$

where

$$T(N,k,m,n) = M_N(2k)\frac{N!(N+2k)!\left(\frac{1}{2}N+\frac{1}{2}n+k\right)!\left(\frac{1}{2}N+\frac{1}{2}n+k+\frac{1}{2}\right)!}{(N-n)!(N+k)!^2} \times \frac{2^{N-n}\left(-\frac{1}{2}N+\frac{1}{2}n\right)_m\left(-\frac{1}{2}N+\frac{1}{2}n+\frac{1}{2}\right)_m(k+n)!}{m!n!(2k+n)!\left(n+k+\frac{1}{2}+m\right)!}.$$

Observe that for fixed k, m, n with $\mathrm{Re}(k) > -1/2$,

$$T(N,k,m,n) = \frac{G^2(k+1)}{G(2k+1)}\frac{(k+n)!(k+n+m)!}{m!n!(2k+n)!(2k+2n+2m+1)!}N^{(k+1)^2+2n+2m}\left(1+\mathrm{O}\left(\frac{1}{N}\right)\right)$$

and so

$$\mathbb{E}_N\{|Z(0)|^{2k}|Z(y/N)|^2\} \sim \frac{G^2(k+1)}{G(2k+1)}N^{(k+1)^2} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{(k+n)!(k+n+m)!}{m!n!(2k+n)!(2k+2n+2m+1)!}y^{2m+2n}$$

in the sense that for each fixed integer h, the coefficient of y^h on the Taylor expansion of the left-hand side converges to that of the right-hand side as $N \to \infty$.

Finally we show that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{(k+n)!(k+n+m)!}{m!n!(2k+n)!(2k+2n+2m+1)!}y^{2m+2n} = \sum_{p=0}^{\infty}\frac{k(k-1+p)!(k+p)!}{p!(2k+p)!(2k+1+2p)!}(-1)^p y^{2p}.$$

This can be proven by comparing the coefficients of y^{2p}. That is, we wish to show for all integer $p \geqslant 0$,

$$\sum_{n=0}^{p}\frac{(-1)^{p-n}(k+n)!(k+p)!}{(p-n)!n!(2k+n)!(2k+2p+1)!} = \frac{(-1)^p k(k-1+p)!(k+p)!}{p!(2k+p)!(2k+1+2p)!}.$$

This is equivalent to showing

$$\sum_{n=0}^{p}\frac{(-1)^n(k+n)!p!(2k+p)!}{k(p-n)!n!(2k+n)!(k-1+p)!} = 1 \tag{8}$$

for all integer $p \geqslant 0$, and this we shall do by creating the Wilf–Zeilberger pair [17]. Denote the summand in (8) by $F(p, n)$, and observe that

$$F(p+1, n) - F(p, n) = G(p, n+1) - G(p, n) \tag{9}$$

where

$$G(p, n) = \frac{(2k+n)n}{(n-p-1)(k+p)} F(p, n)$$

(the $(2k+n)n/(n-p-1)(k+p)$ being calculated by Zeilberger's algorithm). Summing both sides of (9) over n, we see that the right-hand side telescopes to zero, which shows that the left-hand side of (8) must be a constant, independent of p. Putting $p = 0$, direct calculation shows that the constant is 1. □

3. Conjecture about the zeta function

Conjecture 5. *For fixed k subject to* $\mathrm{Re}(k) > -1/2$,

$$\frac{1}{N(T)} \sum_{0<\gamma_n \leqslant T} \left| \zeta\left(\frac{1}{2} + \mathrm{i}\,(\gamma_n + \alpha/L)\right) \right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} a(k) F_k(2\pi\alpha) \left(\log \frac{T}{2\pi}\right)^{k^2}$$

as $T \to \infty$, *uniformly in* α *for* $|\alpha| \leqslant L$, *where* $L = \frac{1}{2\pi} \log \frac{T}{2\pi}$ *is the density of zeros of height* T, $G(\cdot)$ *is the Barnes G-function,* $a(k)$ *is given by (1) and* $F_k(2\pi\alpha)$ *is given in theorem 3.*

If this conjecture is true, then we are able to prove conjecture 2 and a variant of the Keating–Snaith conjecture (conjecture 1).

Corollary 5.1. *If conjecture 5 is true, then*

$$\frac{1}{N(T)} \sum_{0<\gamma_n \leqslant T} \left| \zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_n\right) \right|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a(k) \left(\log \frac{T}{2\pi}\right)^{k(k+2)}.$$

Proof. By the definition of differentiation,

$$\left| \zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_n\right) \right|^{2k} = L^{2k} \lim_{a \to 0} \frac{\left|\zeta\left(\frac{1}{2} + \mathrm{i}\left(\gamma_n + \frac{\alpha}{L}\right)\right)\right|^{2k}}{\alpha^{2k}}.$$

From (5) we have

$$\lim_{\alpha \to 0} \frac{F(2\pi\alpha)}{\alpha^{2k}} = (2\pi)^{2k} \frac{k!k!}{(2k)!(2k+1)!}$$

so applying conjecture 5 and using uniformity to swap the $\alpha \to 0$ and $N \to \infty$ limits, we have

$$\begin{aligned} \frac{1}{N(T)} \sum_{0<\gamma_n \leqslant T} \left| \zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_n\right) \right|^{2k} &\sim \frac{G^2(k+1)}{G(2k+1)} a(k) L^{2k} \frac{(2\pi)^{2k} k!k!}{(2k)!(2k+1)!} \left(\log \frac{T}{2\pi}\right)^{k^2} \\ &= \frac{G^2(k+2)}{G(2k+3)} a(k) \left(\log \frac{T}{2\pi}\right)^{k(k+2)} \end{aligned}$$

as required. □

Corollary 5.2. *From conjecture 5 it follows that for* $\beta > 0$ *fixed,*

$$\frac{1}{N(T)} \sum_{0<\gamma_n \leqslant T} \left| \zeta\left(\frac{1}{2} + \mathrm{i}(\gamma_n + \beta)\right) \right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} a(k) \left(\log \frac{T}{2\pi}\right)^{k^2}. \tag{10}$$

Proof. The asymptotics as $z \to \infty$ of the spherical Bessel function (see chapters 9 and 10 of [14]) are

$$j_n(z) \sim \begin{cases} \frac{1}{z}(-1)^{(n+1)/2}\cos z & \text{if} \quad n \text{ is odd} \\ \frac{1}{z}(-1)^{n/2}\sin z & \text{if} \quad n \text{ is even} \end{cases}$$

and putting this into (3) we have

$$\lim_{\alpha\to\infty} F_k(2\pi\alpha) = 1.$$

Setting $\alpha = L\beta$ in conjecture 5 completes the proof. □

Remark. Note that corollary 5.2 can be thought of as a variant of conjecture 1. This is because one expects the mean of $|\zeta(1/2+\mathrm{i}t)|^{2k}$ to be independent of the average taken, and the Keating–Snaith conjecture is a result about the continuous mean, whereas corollary 5.2 is a result about a discrete mean. To see this, recall that the zeros get denser higher up the critical line, and so if $\beta > 0$ is fixed and γ_n is random, one might expect $\zeta\left(\frac{1}{2}+\mathrm{i}(\gamma_n+\beta)\right)$ to be random (whereas, if β was small, it would be highly influenced by the fact that $\zeta(1/2+\mathrm{i}\gamma_n) = 0$). The left-hand side of (10) averages this, and thus acts as a discrete mean of $|\zeta(1/2+\mathrm{i}t)|^{2k}$.

3.1. Comparison with the zeta function

Gonek [9] showed that if the Riemann hypothesis is true then

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leqslant T}\left|\zeta\left(\frac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^2 \sim \left(1-\left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)^2\right)\log\frac{T}{2\pi}$$

uniformly in α for $|\alpha| \leqslant L/2$, which is in perfect agreement with conjecture 5 when $k = 1$.

There is no proof of the conjecture for $k = 2$ (unlike conjecture 1 which is proven for $k = 1$ and 2). But there are theorems along the lines of conjecture 5 for $k = 2$.

Theorem 6 (Conrey, Ghosh and Gonek [10])**.** *Assume GRH and let*

$$A(s) = \sum_{n\leqslant x} n^{-s} \qquad \text{where} \quad x = \left(\frac{T}{2\pi}\right)^{\eta}$$

for some $\eta \in \left(0, \frac{1}{2}\right)$*. Then,*

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leqslant T}\left|\zeta A\left(\frac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^2 \sim \frac{6}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(2\pi\alpha)^{2j+2}}{(2j+5)!}$$
$$\times\left(-\eta^2+\frac{1}{3}(2j+5)\eta^3-\frac{2j+5}{j+3}\eta^{2j+6}+\eta^{2j+7}+\eta^2(1-\eta)^{2j+5}\right)\left(\log\frac{T}{2\pi}\right)^4$$

uniformly for bounded α*.*

(We have slightly changed notation from [10], to be consistent with our definition of $L = \frac{1}{2\pi}\log\frac{T}{2\pi}$.)

Putting $\eta = 1$ in the above (which, as it stands, is not allowed under the conditions of the theorem) then $A\left(\frac{1}{2}+\mathrm{i}t\right) = \zeta\left(\frac{1}{2}+\mathrm{i}t\right) + \mathrm{O}(t^{-1/2})$, and we have

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leqslant T}\left|\zeta^2\left(\frac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^2 \sim \frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(2\pi\alpha)^{2j+2}}{(2j+6)!}(2j^2+5j)\left(\log\frac{T}{2\pi}\right)^4.$$

Note that

$$\frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(2\pi\alpha)^{2j+2}}{(2j+6)!}(2j^2+5j)$$
$$= \frac{1}{12}a(2)\frac{(2\pi^2\alpha^2-3)\sin^2(\pi\alpha)+3\pi\alpha\sin(2\pi\alpha)+(\pi\alpha)^4-3(\pi\alpha)^2}{(\pi\alpha)^4}$$

which is what is predicted in conjecture 5.

That is, from a purely number theoretical calculation involving no random matrix theory, we have

Conjecture 7. *Assuming that $\eta = 1$ is permissible in theorem 6 then*

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leqslant T}\left|\zeta\left(\frac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^4 \sim \frac{1}{2\pi^2}F_2(2\pi\alpha)\left(\log\frac{T}{2\pi}\right)^4$$

where $F_2(2x)$ is given in (6).

So, following the proof of corollary 5.1, we may deduce

Corollary 7.1. *If conjecture 7 is true then*

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leqslant T}\left|\zeta'\left(\frac{1}{2}+\mathrm{i}\gamma_n\right)\right|^4 \sim \frac{1}{1440\pi^2}\left(\log\frac{T}{2\pi}\right)^8.$$

Note that this is the same answer that one gets from putting $k = 2$ into conjecture 2.

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